Cramer's Rule for a 3×3 Matrix

Problem: Solve the system $\begin{array}{c} 2x-3y+2z=10 \\ x+3y+4z=14 \\ 3x-y+z=9 \end{array}$ using Cramer's Rule.

Solution: We have the coefficient matrix $\begin{bmatrix} 2 & -3 & 2 \\ 1 & 3 & 4 \\ 3 & -1 & 1 \end{bmatrix}$, variable matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and constant matrix $\begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$. First $\begin{vmatrix} 2 & -3 & 2 \\ 1 & 3 & 4 \\ 3 & -1 & 1 \end{vmatrix} = -39$. Next, to find the value of x we substitute the constant matrix $\begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$ in the first column of the coefficient matrix. So that we have, $\begin{bmatrix} 10 & -3 & 2 \\ 14 & 3 & 4 \\ 9 & -1 & 1 \end{bmatrix}$. Then we find the determinant $\begin{vmatrix} 10 & -3 & 2 \\ 14 & 3 & 4 \\ 9 & -1 & 1 \end{vmatrix} = -78$. So that $x = \dfrac{\begin{vmatrix} 10 & -3 & 2 \\ 14 & 3 & 4 \\ 9 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 2 \\ 1 & 3 & 4 \\ 3 & -1 & 1 \end{vmatrix}} = \dfrac{-78}{-39} = 2$. Similarly, to find y we substitute the constant matrix in the second column of the coefficient matrix $\begin{bmatrix} 2 & 10 & 2 \\ 1 & 14 & 4 \\ 3 & 9 & 1 \end{bmatrix}$. Then we find $\begin{vmatrix} 2 & 10 & 2 \\ 1 & 14 & 4 \\ 3 & 9 & 1 \end{vmatrix} = 0$. So that, $y = \dfrac{\begin{vmatrix} 2 & 10 & 2 \\ 1 & 14 & 4 \\ 3 & 9 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 2 \\ 1 & 3 & 4 \\ 3 & -1 & 1 \end{vmatrix}} = \dfrac{0}{-39} = 0$. Finally, to find z we substitute the constant matrix in the third column of the coefficient matrix $\begin{bmatrix} 2 & -3 & 10 \\ 1 & 3 & 14 \\ 3 & -1 & 9 \end{bmatrix}$. Then we find $\begin{vmatrix} 2 & -3 & 10 \\ 1 & 3 & 14 \\ 3 & -1 & 9 \end{vmatrix} = -117$. So that, $z = \dfrac{\begin{vmatrix} 2 & -3 & 10 \\ 1 & 3 & 14 \\ 3 & -1 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 2 \\ 1 & 3 & 4 \\ 3 & -1 & 1 \end{vmatrix}} = \dfrac{-117}{-39} = 3$. Thus our solution is $(2,0,3)$

Problem: Use augmented matrices to find the solution to the above system.

Solution: We have,

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 10 \\ 1 & 3 & 4 & 14 \\ 3 & -1 & 1 & 9 \end{array}\right] \Rightarrow R_1 := R_1 - R_2 \left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 1 & 3 & 4 & 14 \\ 3 & -1 & 1 & 9 \end{array}\right] \Rightarrow R_2 := R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 9 & 6 & 18 \\ 3 & -1 & 1 & 9 \end{array}\right] \Rightarrow R_3 := R_3 - 3R_1 \left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 9 & 6 & 18 \\ 0 & 17 & 7 & 21 \end{array}\right] \Rightarrow R_3 := R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 9 & 6 & 18 \\ 0 & 8 & 1 & 3 \end{array}\right] \Rightarrow R_2 := R_2 - R_3 \left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 1 & 5 & 15 \\ 0 & 8 & 1 & 3 \end{array}\right] R_3 := R_3 - 8R_2$$

$$\left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 1 & 5 & 15 \\ 0 & 0 & -39 & -117 \end{array}\right] \Rightarrow R_3 := \frac{1}{-39} R_3 \left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 1 & 5 & 15 \\ 0 & 0 & 1 & 3 \end{array}\right] R_2 := R_2 - 5R_3$$

$$\left[\begin{array}{ccc|c} 1 & -6 & -2 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{array}{l} R_1 := R_1 + 6R_2 \\ R_1 := R_1 + 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus our solution is $(2,0,3)$.